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Greedy algorithms for a class of knapsack problems with binary weights

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Abstract

In this article we identify a class of two-dimensional knapsack problems and related three-criteria unconstrained combinatorial optimization problems that can be solved in polynomial time by greedy algorithms. For the latter problem, the proposed algorithm explores the connectedness of the set of efficient solutions. Extensive computational results show that this approach can solve the three-criteria problem up to one million items in half an hour.

1 Introduction

The 0/1 multidimensional knapsack problem is a classical NP-hard problem with many applications and for which several theoretical results are known (see Weingartner and Ness [14], Kellerer *et al.* [9]). Due to its hardness, exact algorithms are only able to solve small to medium sized instances. For this reason, many heuristic procedures have been proposed in the literature [1, 5, 11, 12, 13]. By transforming the m constraints into m minimizing criteria, we obtain a special case of the multicriteria unconstrained combinatorial optimization problem [3]. This problem is also NP-hard and to the best of our knowledge no algorithm has been proposed to solve it.

In this article, we consider the 0/1 m-dimensional knapsack problem with binary weight coefficients and the $(m+1)$-criteria unconstrained optimization problem with m binary criteria coefficients. We show that for $m = 2$, the problems above can be solved to optimality in polynomial time by following a simple greedy strategy.

Two additional aspects are worthwhile noting. First, the greedy algorithm for the three-criteria problem with binary criteria coefficients provides a constructive proof that the set of the efficient solutions for this problem is connected according to a combinatorial definition of neighborhood [6, 7]. We remark that this is the first positive and non-trivial result of connectedness in multicriteria combinatorial optimization. In general, it has been shown that the efficient set is not connected for many of these problems [4, 7]. Second, the cardinality of the nondominated set is bounded by a polynomial function of the number of items, which is often not the case in multicriteria combinatorial optimization [2]. The greedy algorithm proposed in this article solves the three-criteria problem in polynomial time and takes constant amount of time to find each efficient solution after a pre-processing step. Our numerical experiments indicate that this approach is very efficient in practice.

The article is organized as follows. In Section 2, we introduce the notation and the problems. Furthermore we present the pre-processing phase that is common to all algorithms described in this article. The greedy algorithm for the 0/1 two-dimensional problem with equality constraints is presented in Section 3. The algorithms for the three-criteria problem and for the 0/1 two-dimensional problem with inequality constraints are presented in Sections 4 and 5, respectively. We conclude in Section 6.

2 Notation and Pre-Processing

Let $j \in \{1, \ldots, m\}$. We assume that there exists a set N of n items available. We denote the profit of each item $s \in N$ and its weight at dimension j by $p(s)$ and $w^j(s)$, respectively. The knapsack capacity at dimension j is denoted by c^j. Let $S \subseteq N$ be a subset of items (called a *knapsack* in the following). We denote by $p(S) = \sum_{s \in S} p(s)$ and $w^j(S) = \sum_{s \in S} w^j(s)$ the total profit and the total weight at dimension j of the items in knapsack S, respectively. In our particular case, we assume that $w^j(s)$ can only take binary values for all items $s \in N$.

Definition 2.1 (The 0/1 m-dimensional knapsack problem with binary weights (m-KP$_\leq$)) *Given a finite set N, for each $s \in N$ a profit $p(s) \in \mathbb{Z}^+$ and a weight $w^j(s) \in \{0, 1\}$, and a non-negative integer c^j, find a subset $S \subseteq N$ such that $p(S)$ is maximal and $w^j(S) \leq c^j$, for $j = 1, \ldots, m$.*

When $m = 1$, Problem (m-KP$_\leq$) simplifies to a sequential knapsack problem with divisible weights, which is solvable in $O(n \log n)$ time [8]. We introduce the following special case of (m-KP$_\leq$).

Definition 2.2 (The 0/1 m-dimensional knapsack problem with equality constraints and binary weights (m-KP$_=$)) *Given a finite set N, for each $s \in N$ a profit $p(s) \in \mathbb{Z}^+$ and a weight $w^j(s) \in \{0, 1\}$, and a non-negative integer c^j, find a subset $S \subseteq N$ such that $p(S)$ is maximal and $w^j(S) = c^j$, for $j = 1, \ldots, m$.*

If we transform the m constraints of Problems (m-KP$_=$) and (m-KP$_\leq$) into m criteria to minimize, we obtain a variant of the multicriteria unconstrained combinatorial optimization problem [3] that is defined as follows.

Definition 2.3 (The $(m+1)$-criteria unconstrained combinatorial optimization problem with m binary weights (m-MP)) *Given a finite set N, for each $s \in N$ a profit $p(s) \in \mathbb{Z}^+$ and a weight $w^j(s) \in \{0, 1\}$, find a subset $S \subseteq N$ such that $p(S)$ is maximal and $w^j(S)$ is minimal, for $j = 1, \ldots, m$.*

We define the notion of optimality for Problem (m-MP) as follows. Let $\mathcal{N}$ denote the set of feasible knapsacks. The image of the set $\mathcal{N}$ when using the $m+1$ criteria forms a set of points in the criteria space, denoted here by $\mathcal{Z} \subseteq \mathbb{N}^{m+1}$. We say that a feasible knapsack S *dominates* another feasible knapsack S' if and only if $p(S) \geq p(S')$ and $w^j(S) \leq w^j(S')$ for $j = 1, \ldots, m$, with at least one strict inequality; if strict inequality holds for all $m+1$ components, then S *strictly-dominates* S'. If there is no feasible knapsack that dominates S, then we say that S is an *efficient knapack*; in case there exists no feasible knapsack that strictly-dominates S, then this knapsack becomes *weakly-efficient*. The set of all efficient knapsacks is denoted by $\mathcal{N}^E$. An efficient knapsack is called *supported efficient*, if it is a minimizer of a non-trivial weighted sum problem with the three objectives $-p(S)$, $w^1(S)$ and $w^2(S)$. We say that a vector $Z \in \mathcal{Z}$ is *non-dominated* if there is some efficient knapsack $S \in \mathcal{N}^E$ such that Z is the criteria vector of S for Problem (m-MP). We denote the set of all non-dominated vectors, the *non-dominated set*, by $\mathcal{Z}^{ND}$.

We recall that a given efficient knapsack of Problem (m-MP) corresponds to an optimal knapsack for Problem (m-KP$_\leq$) for appropriate chosen, non-negative integers $c^1, \ldots, c^m$. Moreover, if there exists an optimal knapsack for Problem (m-KP$_\leq$) with non-negative integers $c^1, \ldots, c^m$, then this knapsack is at least weakly-efficient for Problem (m-MP)[3]. Note that if there exists an optimal knapsack for Problem (m-KP$_=$) with non-negative integers $c^1, \ldots, c^m$, this knapsack may not even be weakly-efficient for (m-MP).

In this article, we are particularly interested in the two-dimensional case ($m = 2$) of Problems (m-KP$_\leq$) and (m-KP$_=$). For the latter problem, we denote the set of all optimal knapsacks for the constraint (c^1, c^2) by $\mathcal{S}(c^1, c^2)$; if $c^1 = 0$ (or $c^2 = 0$) we call $\mathcal{S}(0, c^2)$ (or $\mathcal{S}(c^1, 0)$) a *basis* with respect to c^2 (or c^1). Moreover, $\rho(c^1, c^2)$ denotes the optimal profit value for this problem with constraint (c^1, c^2), i.e. $\rho(c^1, c^2) = p(S)$ where $S \in \mathcal{S}(c^1, c^2)$.

For the Problem (m-MP) with $m = 2$, we consider the problem of finding set $\mathcal{Z}^{ND}$ since to find all efficient knapsacks is an intractable task (cf. [9]); if $p(s) = w^j(s) = 1$ and $c^j = n - 1$, for all $s \in N$ and $j = 1, \ldots, m$, then there exists $2^n - 1$ efficient knapsacks.

For the sake of the explanation, we give a geometric interpretation of the image of all feasible solutions for the two weight criteria w^1 and w^2 as a set

$$\mathcal{G} := \left\{ \left(w^1(S), w^2(S)\right),\, S \subseteq N \right\}$$

that forms an hexagonal grid in the $\mathbb{R}^2$-plane (cf. Figure 1). Note that $|\mathcal{G}| = O(n^2)$ and that this is a strict upper bound on the cardinality of the non-dominated set as shown in the following example.

Example 2.4 Let $N = \{s_1, s_2, s_3\}$, $w(s_1) = (1, 0), w(s_2) = (0, 1), w(s_3) = (1, 1)$, and $p(s) = w^1(s) + w^2(s)$, for all $s \in N$. Then, $\mathcal{Z}^{ND} = \{(0,0,0),\ (0,1,1),\ (1,0,1),\ (1,1,2)\}$ and $|\mathcal{G}| = |\mathcal{Z}^{ND}|$.

In the following sections, we present greedy algorithms to solve the three problems defined above. After a pre-processing phase, the algorithm solves Problem (2-KP$_=$) by inserting items into the knapsack according to a given sequence of items. The non-dominated set of Problem (2-MP) is found by iteratively solving the previous problem for several constraints. Finally, Problem (2-KP$_\leq$) is solved based on the results for Problem (2-MP).

The following pre-processing step is common to all algorithms presented in this article. It consists of partitioning the set of items and sorting its elements with respect to their profit values. Without loss of generality, we assume that $w^1(s) + w^2(s) \geq 1$ for all items s of the problem. Note that items with null weights will only augment the profit value

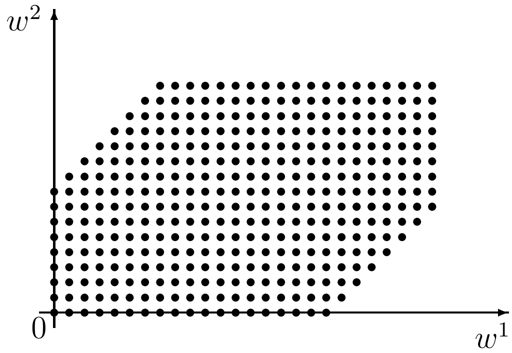

Figure 1: *The hexagonal grid $\mathcal{G}$ in the $\mathbb{R}^2$-plane.*

of a knapsack. Therefore, we can remove those items, store the sum of their profits and solve the problem for the remaining ones.

We partition the set of items according to their weights $(w^1(s), w^2(s))$ for all items s and obtain three different sets where all elements in a set have the weights $(1, 0)$, $(1, 0)$ and $(1, 1)$, respectively. We denote these sets by R, U and D, respectively, and their cardinalities by n_R, n_U and n_D. Without loss of generality, we assume that $n_U \leq n_R$. Next, we sort the elements of each set in non-increasing order of the profit values. We store the profit values of these items in the sequences $r = (r_1, r_2, \dots, r_{n_R})$, $u = (u_1, u_2 \dots, u_{n_U})$ and $d = (d_1, d_2, \dots, d_{n_D})$.

In the following sections, we will interleave between sequences of items and sequences of profits. The correspondence between an item in the sequence U, R or D and its profit in the sequence u, r or d should be clear from the context.

3 A greedy algorithm for Problem (2-KP$_=$)

The greedy algorithm described in this section returns an optimal knapsack for Problem (2-KP$_=$) for an arbitrary constraint (c^1, c^2) in $\mathcal{G}$. The algorithm starts by finding an optimal knapsack for a given basis and proceeds by filling it with items taken from sets U, R and D based on the decomposition

$$\begin{pmatrix} w^1(S) \\ w^2(S) \end{pmatrix} = \left(c^1 - c^2\right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c^2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} \tag{1}$$

for $c^1 \geq c^2$ and a similar decomposition for $c^2 \geq c^1$, respectively.

For the sake of the explanation, we call as *super-item* a pair of items where one is taken from set R and the other is taken from set U. A super-item has a weight $(1, 1)$ and its profit is the sum of the profits of the two items.

For the remaining results of this section, we state the following remark.

Remark 3.1 *For a given constraint $(c^1, c^2) \in \mathcal{G} \setminus \{(0, 0)\}$, there is at least one optimal knapsack that contains the first items from set U, R or D.*

The first lemma states that it is easy to find the optimal profit value of knapsacks in a basis.

Lemma 3.2 *Let $c^1 \in \{1,\ldots,n_R\}$ and $c^2 \in \{1,\ldots,n_U\}$. Then, for all knapsacks $S \in \mathcal{S}(c^1,0)$ and $S' \in \mathcal{S}(0,c^2)$ it holds*

$$p(S) = \sum_{i=1}^{c^1} r_i \quad \text{and} \quad p(S') = \sum_{i=1}^{c^2} u_i.$$

Starting from an optimal knapsack in a basis, we establish the connection between these knapsacks and optimal knapsacks in $\mathcal{S}(c^1,c^2)$.

Lemma 3.3 *Let $(c^1,c^2) \in \mathcal{G}$ with $0 < c^2 < c^1$. Then, there exists an optimal knapsack $S \in \mathcal{S}(c^1,c^2)$ that contains all items of an optimal knapsack $S' \in \mathcal{S}(c^1 - c^2, 0)$.*

Clearly, a similar result to Lemma 3.3 can be obtained for the case that $0 < c^1 < c^2$.

Lemma 3.3 suggests a greedy algorithm to solve Problem (2-KP$_=$) for a given $(c^1,c^2) \in \mathcal{G}$. Assume that $0 < c^2 \leq c^1$. First, fill the knapsack with the first $c^1 - c^2$ items from R. Then repeat the following procedure c^2 times:

(i) Select the three items with the largest profit in R, U and D, respectively, that are not in the knapsack; let the two items from R and U correspond to a super-item.

(ii) From the item of D or the super-item of R and U, insert the one with the largest profit into the knapsack.

Let $\bar{D}$ denote the sequence of these last c^2 (super-)items. The following theorem states that the application of the above given procedure results in an optimal knapsack for a given instance of (2-KP$_=$) where $0 < c^2 \leq c^1$.

Theorem 3.4 *Let $(c^1,c^2) \in \mathcal{G}$ such that $0 < c^2 \leq c^1$, and let S denote the knapsack that includes the first $c^1 - c^2$ items from R and the c^2 (super-)items from $\bar{D}$. Then $S \in \mathcal{S}(c^1,c^2)$.*

Proof. According to the description above, S is a feasible knapsack and satisfies the decompostion (1). Now, assume that S is not optimal, i.e. there exists another feasible knapsack S' such that $p(S') > p(S)$. According to the construction of S as well as the result from Remark 3.1 and Lemma 3.3, we may assume that S' includes the first $c^1 - c^2$ elements of R. Therefore, the weights of the remaining items in S' must sum up to (c^2,c^2), since otherwise S' would not be a feasible knapsack. However, this is only possible if the cardinality of items from U and the cardinality of additional items from R coincide since they do not augment the value of the two weight coefficients simultaneously as all the items in D do. Hence, S' must contain a combination of items (aside from the first $c^1 - c^2$ items of R) whose total profit is equal to the sum of the c^2 profit values of (super-)items in $\bar{D}$. However, this means that $p(S') = p(S)$, which contradicts the assumption that S is not in $\mathcal{S}(c^1,c^2)$. $\square$

Using the same reasoning as above, we can construct an optimal knapsack for the case where $0 < c^1 \leq c^2$. This optimal knapsack contains the first $c^2 - c^1$ items of U and the c^2 items of the sequence $\bar{D}$, which corresponds to the sequence of items from D and appropriate combined super-items of R and the remaining items in U.

Note that since sequences R, U and D are sorted according to the profits, we can find the optimal knapsack in linear time after the pre-processing phase. The algorithm takes $O(n \log n)$ time due to the sorting step at the pre-processing phase.

4 A greedy algorithm for Problem (2-MP)

Based on the results of the previous section, we can derive a straight-forward algorithm that finds the non-dominated set for Problem (2-MP) in polynomial time: First, call the greedy algorithm to solve Problem (2-KP$_=$) for every constraint $(c^1, c^2) \in \mathcal{G}$ (see Section 3) and store each optimal profit found and corresponding weight vector as a tuple; then, remove the dominated tuples. Clearly, the remaining set of tuples corresponds to the non-dominated set of Problem (2-MP). Since the removal of dominated tuples can be performed in $O(n \log n)$ [10], this algorithm has $O(n^3 \log n)$ time complexity. In this section we present an improvement on the algorithm above that reduces the time complexity to $O(n^2)$. The resulting algorithm is optimal in terms of upper bound time complexity.

The remaining parts of this section are organized as follows: After proving important dominance relations for different pairs of $(c^1, c^2) \in \mathcal{G}$, we derive a strict lower bound on the cardinality of the non-dominated set. Using these results we show that we can find the non-dominated set of Problem (2-MP) without even applying the filtering step to remove dominated knapsacks at the end.

In the following we will focus on the case that $0 < c^2 \leq c^1$ and only mention equivalent results for the case that $0 < c^1 \leq c^2$ briefly. Hence, let $0 < c^2 \leq c^1$, let $c = c^1 - c^2$ and let $\lambda = \min\{n_R - c, n_U\}$. For a given basis $\mathcal{S}(c, 0)$, $c \in \{0, \ldots, n_R\}$, let $\bar{D}^c$ denote the sequence of λ (super-)items that are chosen according to the greedy algorithm described in Section 3. Moreover, we store the profits of the elements in $\bar{D}^c$ in the sequence $\bar{d}^c$ and the profits of the super-items in the sequence $\tilde{d}^c$. In this case we say that these sequences *correspond* to the given basis c.

For the remaining results of this section, we introduce the following corollary.

Corollary 4.1 *Let $(c^1, c^2) \in \mathcal{G}$ such that $c^1 = n_R + n_D$. Then there exists a sequence $\{S^i\}_i$ of knapsacks such that $S^i \in \mathcal{S}(c^1 - c^2 + i, i)$ for $i = 0, \ldots, c^2$ and S^i and S^{i+1} differ in exactly one (super-)item in the sequence $\bar{D}^{c^1 - c^2}$.*

Proof. The proof follows directly from Theorem 3.4. Let $S^0 \in \mathcal{S}(c^1 - c^2, 0)$ be the knapsack that contains the first $c^1 - c^2$ items of R and let S^i contains all items from S^0 and the first i (super-)items in sequence $\bar{D}^{c^1 - c^2}$ for $i \in \{1, \ldots, c^2\}$. Then, $S^i \in \mathcal{S}(c^1 - c^2 + i, i)$ for $i = 0, \ldots, c^2$ and S^i and S^{i+1} differ in exactly one (super-)item in the sequence $\bar{D}^{c^1 - c^2}$. □

Clearly, a similar result holds for the case that $c^2 = n_U + n_D$.

Corollary 4.1 suggests that Problem (2-KP$_=$) can be solved for several constrained values by starting from a knapsack in a basis and subsequently adding c^2 items from the sequence $\bar{D}^{c^1 - c^2}$ for $c^1 = n_R + n_D$. By repeating this procedure for each basis, we obtain an algorithm that finds the profit values of Problem (2-KP$_=$) for all constraints in the grid $\mathcal{G}$ in $O(n^2)$ time. Then, the non-dominated set for Problem (2-MP) can be found in $O(n^2)$.

In the remaining part of this section, we will show that we may not need to consider the complete sequence $\bar{D}^c$ as defined above and that the removal of dominated knapsacks does not even need to be performed. For this purpose, we need to split the grid $\mathcal{G}$ into three sectors $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ defined as follows: $\mathcal{G}_1$ corresponds to the points of the grid that do not lie under the line segment connecting the points $(0, 0)$ and $(n_U + n_D, n_U + n_D)$; $\mathcal{G}_2$ consists of all points in the grid between the line segments connecting the points $(0, 0)$

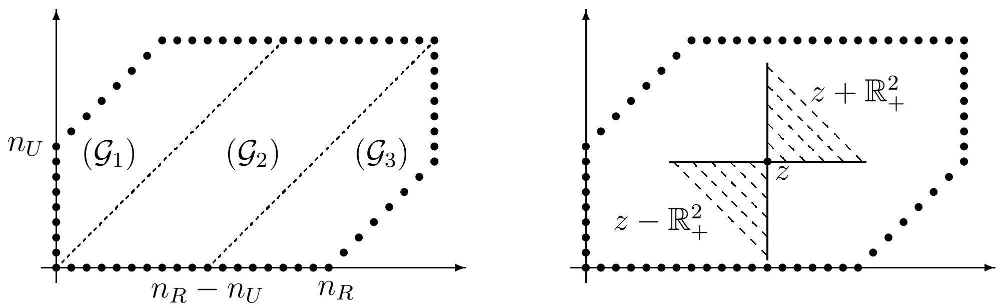

Figure 2: *Sectors and Dominance.*

and (n_U+n_D, n_U+n_D) and the points $(n_R-n_U, 0)$ and (n_R+n_D, n_U+n_D), respectively; the remaining points of the grid form $\mathcal{G}_3$ (see also the left-hand side in Figure 2). Note that all integer points on the border of two sectors belong to both sectors and that it is assumed that $n_U \leq n_R$.

4.1 Dominance Relations in $\mathcal{G}$

In the following we establish *dominance relations* among points in $\mathcal{G}$. For a constraint $(c^1, c^2) \in \mathcal{G}$ we know that a knapsack $S \in \mathcal{S}(c^1, c^2)$ can be dominated by any other knapsack in $\mathcal{S}((c^1,c^2)-\mathbb{R}^2) \cap \mathcal{G})$ and that S potentially dominates knapsacks in $\mathcal{S}((c^1,c^2)+\mathbb{R}^2) \cap \mathcal{G})$ by definition of Problem (2-MP) (cf. also Figure 2). In addition, while knapsack S can never dominate (or be dominated by) knapsacks in $\mathcal{S}(c^1+1, c^2+1)$ (or in $\mathcal{S}(c^1-1, c^2-1))$ by the construction of the sequences in Corollary 4.1, this can be the case for all knapsacks in $\mathcal{S}(c^1+1, c^2)$ or in $\mathcal{S}(c^1, c^2+1)$ (or in $\mathcal{S}(c^1-1, c^2)$ or in $\mathcal{S}(c^1, c^2-1)$). In the following we will focus on these cases.

To simplify the notation, we say that there exists *horizontal dominance*, *to the right* or *from the left* when an optimal knapsack to Problem (2-KP$_=$) with constraint $(c^1, c^2) \in \mathcal{G}$ dominates (or is dominated by) his right (left) neighbor, i.e. an optimal knapsack to Problem (2-KP$_=$) with constraint $(c^1+1, c^2) \in \mathcal{G}$ (or $(c^1-1, c^2) \in \mathcal{G}$, respectively). Furthermore, we say that we have *vertical dominance*, *to the top* or *from the bottom* whenever an optimal knapsack to Problem (2-KP$_=$) with constraint $(c^1, c^2) \in \mathcal{G}$ dominates (or is dominated by) his top (bottom) neighbor, i.e. an optimal knapsack to Problem (2-KP$_=$) with constraint $(c^1, c^2+1) \in \mathcal{G}$ (or $(c^1, c^2-1) \in \mathcal{G}$, respectively). We will show in the following that vertical dominance can apply in $\mathcal{G}_1$ and $\mathcal{G}_2$ while horizontal dominance is not possible in these sectors. For Sector $\mathcal{G}_3$ we will prove that horizontal dominance is possible but never vertical dominance.

We start by introducing two lemmas that will be useful for deriving the main results about vertical and horizontal dominance in the three sectors. Starting from a knapsack $S^0 \in \mathcal{S}(0, c)$ where $c \in \{0, \ldots, n_U\}$ we have that

$$\rho(n_U+n_D-c, n_U+n_D) = \sum_{i=1}^{n_U-c} r_i + \sum_{i=1}^{n_U} u_i + \sum_{i=1}^{n_D} d_i, \tag{2}$$

in $\mathcal{G}_1$, according to Corollary 4.1. For $c \in \{0, \ldots, n_R-n_U\}$ and $S^0 \in \mathcal{S}(c, 0)$ Corollary 4.1 provides that

$$\rho(n_U + n_D + c, n_U + n_D) \;=\; \sum_{i=1}^{n_U+c} r_i + \sum_{i=1}^{n_U} u_i + \sum_{i=1}^{n_D} d_i, \tag{3}$$

in $\mathcal{G}_2$, whereas in $\mathcal{G}_3$ we have that

$$\rho(n_R + n_D, n_R + n_D - c) \;=\; \sum_{i=1}^{n_R} r_i + \sum_{i=1}^{n_R-c} u_i + \sum_{i=1}^{n_D} d_i, \tag{4}$$

where $c \in \{n_R - n_U, \dots, n_R\}$ and $S^0 \in \mathcal{S}(c, 0)$. Since all profits are positive by assumption, it follows from (2), (3) and (4):

Lemma 4.2 *The finite sequences*

$$\{\rho(n_D + i, n_U + n_D)\}_{i=0}^{n_R} \;\; \textit{and} \;\; \{\rho(n_D + n_R, n_D + i)\}_{i=0}^{n_U}$$

are strictly increasing.

Focusing on the Sectors $\mathcal{G}_2$ and $\mathcal{G}_3$ we establish the following relations.

Lemma 4.3 *Let the sequence $\bar{d}^c$ correspond to the basis $\mathcal{S}(c, 0)$, where $c \in \{0, \dots, n_R\}$.*

1. *Let $c \in \{0, \dots, n_R - 1\}$ and $i \in \{1, \dots, n_R - c\}$. Then $\bar{d}^c_k \geq \bar{d}^{c+i}_k$ for all $k \in \{1, \dots, n_D + \min(n_R - (c + i), n_U)\}$.*
2. *Let $c \in \{0, \dots, n_R - n_U - 1\}$. Then $\bar{d}^c_{k+1} \leq \bar{d}^{c+1}_k$ for all $k \in \{1, \dots, n_U + n_D - 1\}$.*
3. *Let $c \in \{n_R - n_U, \dots, n_R - 1\}$. Then $\bar{d}^c_{k+1} \leq \bar{d}^{c+1}_k$ for all $k \in \{1, \dots, n_R + n_D - (c+1)\}$.*

Proof. In the following we prove the three cases above:

(1.) The proof follows immediately from the construction of the sequences $\bar{d}^c$ and $\bar{d}^{c+i}$.

(2.) We distinguish two cases. For the first case we assume that the first element of the sequence $\bar{d}^c$ corresponds to the profit of a super-item. Then, n_D of the remaining $n_U + n_D - 1$ elements of $\bar{d}^c$ coincide with n_D elements of $\bar{d}^{c+1}$ since both sequences contain the n_D profit values of the items in D. In addition, by the construction of the super-items it holds that

$$\tilde{d}^c_{k+1} \;=\; r_{c+k+1} + u_{k+1} \;\leq\; r_{c+k+1} + u_k \;=\; \tilde{d}^{c+1}_k$$

for all $k \in \{1, \dots, n_U - 1\}$. Since the elements of the sequences are sorted in non-increasing order, this implies that $\bar{d}^{c+1}_k \geq \bar{d}^c_{k+1}$ for all $k \in \{1, \dots, n_U + n_D - 1\}$. For the second case, let $\ell > 1$ corresponds to the index of the profit of the first super-item contained in the sequence $\bar{d}^c$. For $k \in \{1, \dots, \ell - 2\}$ it holds that $\bar{d}^c_k = d_k \geq d_{k+1} = \bar{d}^c_{k+1}$ by construction and, in addition, we have that $\bar{d}^c_\ell = \tilde{d}^c_1 = r_{c+1} + u_1 \geq r_{c+2} + u_1 = \tilde{d}^{c+1}_1$. This implies that $\bar{d}^{c+1}_k = d_k$ for $k = 1, \dots, \ell - 1$ and Part (2.) is true at least until $k = \ell$. For the remaining indices we can apply the same reasoning as in the first case taking into account that the remaining elements of the sequence $\bar{d}^c$ and $\bar{d}^{c+1}$ only coincide in the $n_D - (\ell - 1)$ profit values $d_\ell, \dots, d_{n_D}$ of items contained in D. This completes the proof for the second case.

(3.) The proof is similar to the proof of Part (2.).

□

We will use Lemma 4.2 and Lemma 4.3 to derive results for vertical and horizontal dominance between neighbor points in the same sector.

Theorem 4.4 *For Sector $\mathcal{G}_3$ it holds:*

1. *Let $(c^1, c^2) \in \mathcal{G}$ such that $n_R - n_U \le c^1 - c^2 \le n_R - 1$ and $n_R - n_U \le c^1 \le n_R + n_D - 1$. If $S \in \mathcal{S}(c^1, c^2)$ dominates $S' \in \mathcal{S}(c^1+1, c^2)$ then $\bar{S} \in \mathcal{S}(c^1+j, c^2+j)$ also dominates $\bar{S}' \in \mathcal{S}(c^1+j+1, c^2+j)$, where $j \in \{0, \ldots, n_D + n_R - (c^1+1)\}$.*

2. *Let $c \in \{n_R - n_U, \ldots, n_R - 1\}$ and let $S \in \mathcal{S}(c+i+1, i)$ and $S' \in \mathcal{S}(c+i+1, i+1)$, where $i \in \{0, \ldots, n_R + n_D - (c+1)\}$. Then S does not dominate S'.*

Proof. The proof for the two cases above is as follows:

(1.) Let S dominate S', i.e. $p(S) \ge p(S')$. Since according to Lemma 4.3 it holds that $\bar{d}_k^{c^1-c^2} \ge \bar{d}_k^{c^1-c^2+1}$ for all $k \in \{1, \ldots, n_D + n_R - (c^1 - c^2 + 1)\}$, we have that

$$\begin{aligned} p(\bar{S}) &= p(S) + \sum_{k=c^2+1}^{c^2+j} \bar{d}_k^{c^1-c^2} \ge p(S) + \sum_{k=c^2+1}^{c^2+j} \bar{d}_k^{c^1-c^2+1} \\ &\ge p(S') + \sum_{k=c^2+1}^{c^2+j} \bar{d}_k^{c^1-c^2+1} = p(\bar{S}'), \end{aligned}$$

for $j \in \{1, \ldots, n_D + n_R - (c^1+1)\}$. This implies that $\bar{S}$ dominates $\bar{S}'$.

(2.) To prove this case, assume that there exists an index $i \in \{0, \ldots, n_R + n_D - (c+1)\}$ such that $S \in \mathcal{S}(c+i+1, i)$ dominates $S' \in \mathcal{S}(c+i+1, i+1)$. Using the fact that $\bar{d}_k^{c+1} \ge \bar{d}_{k+1}^{c}$ for all $k \in \{1, \ldots, n_R + n_D - (c+1)\}$ from Lemma 4.3 this would imply that $\bar{S} \in \mathcal{S}(n_R + n_D, n_R + n_D - (c+1))$ dominates $\bar{S}' \in \mathcal{S}(n_R + n_D, n_R + n_D - c)$ and hence $\rho(n_R + n_D, n_R + n_D - (c+1)) \ge \rho(n_R + n_D, n_R + n_D - c)$, which is a contradiction to Lemma 4.2.

□

We established that there can exist horizontal dominance but never vertical dominance in Sector $\mathcal{G}_3$. Note that the same reasoning as in Part (1.) of Theorem 4.4 can be applied for Sector $\mathcal{G}_2$ assuming that there is dominance to the right somewhere in this sector. Nevertheless, the next theorem shows that the assumption of dominance to the right is never met in $\mathcal{G}_2$.

Theorem 4.5 *For Sector $\mathcal{G}_2$ it holds:*

1. *Let $c \in \{0, \ldots, n_R - n_U - 1\}$ and let $S \in \mathcal{S}(c+i, i)$ and $S' \in \mathcal{S}(c+i+1, i)$, where $i \in \{1, \ldots, n_U + n_D\}$. Then S does not dominate S'.*

2. *Let $(c^1, c^2) \in \mathcal{G}$ such that $1 \le c^1 - c^2 \le n_R - n_U$ and $0 \le c^2 \le n_U + n_D - 1$. If $S \in \mathcal{S}(c^1, c^2)$ dominates $S' \in \mathcal{S}(c^1, c^2+1)$, then $\bar{S} \in \mathcal{S}(c^1+j, c^2+j)$ also dominates $\bar{S}' \in \mathcal{S}(c^1+j, c^2+j+1)$, where $j \in \{0, \ldots, n_U + n_D - (c^2+1)\}$.*

Proof. The proofs for (1.) and (2.) follow the same line of argument as the proofs of (2.) and (1.) in Theorem 4.4, respectively. To prove (1.), the first part of Lemma 4.3 has to be used, while the second result can be deduced from (2.) of Lemma 4.3. $\square$

For Sector $\mathcal{G}_1$ we briefly state an analogous result to Lemma 4.3.

Lemma 4.6 *Let the sequence $\bar{d}^c$ correspond to the basis $\mathcal{S}(0, c)$, where $c \in \{0, \ldots, n_U\}$.*

1. *Let $c \in \{0, \ldots, n_U - 1\}$ and $i \in \{1, \ldots, n_U - c\}$. Then $\bar{d}^c_k \geq \bar{d}^{c+i}_k$ for all $k \in \{1, \ldots, n_U + n_D - (c + i)\}$.*

2. *Let $c \in \{0, \ldots, n_U - 1\}$. Then $\bar{d}^c_{k+1} \leq \bar{d}^{c+1}_k$ for all $k \in \{1, \ldots, n_U + n_D - (c + 1)\}$.*

Proof. The proofs follow the same idea of the proofs for Part (1.) and Part (3.) in Lemma 4.3. $\square$

We use Lemma 4.6 to derive that there cannot be horizontal dominance in $\mathcal{G}_1$ but that vertical dominance is possible.

Theorem 4.7 *It holds:*

1. *Let $(c^1, c^2) \in \mathcal{G}$ such that $0 \leq c^2 - c^1 \leq n_U - 1$ and $c^2 \leq n_U + n_D - 1$. If $S \in \mathcal{S}(c^1, c^2)$ dominates $S' \in \mathcal{S}(c^1, c^2 + 1)$ then $\bar{S} \in \mathcal{S}(c^1 + j, c^2 + j)$ also dominates $\bar{S}' \in \mathcal{S}(c^1 + j, c^2 + j + 1)$, where $j \in \{0, \ldots, n_U + n_D - (c^2 + 1)\}$.*

2. *Let $c \in \{0, \ldots, n_U - 1\}$ and let $S \in \mathcal{S}(i, c + i + 1)$ and $S' \in \mathcal{S}(i + 1, c + i + 1)$, where $i \in \{0, \ldots, n_U + n_D - (c + 1)\}$. Then S does not dominate S'.*

Proof. The proofs follow the same ideas of the proofs for Theorem 4.5 using Lemma 4.6 instead of Lemma 4.3. $\square$

To summarize, we have shown that there can be vertical dominance in Sectors $\mathcal{G}_1$ and $\mathcal{G}_2$ and there can be horizontal dominance in Sector $\mathcal{G}_3$. We will use these properties to show the following technical theorem which is very important for the next subsections.

Theorem 4.8 *Let $S \in \mathcal{S}(c^1, c^2)$, where $c^1 \cdot c^2 \neq 0$.*

1. *Let $(c^1, c^2) \in \mathcal{G}_1 \cup \mathcal{G}_2$. Then S cannot be dominated by any knapsack S' satisfying $c^2 \geq w^2(S') \geq w^1(S') + c^2 - c^1$. Furthermore if S is dominated, then there exists a knapsack $\bar{S} \in \mathcal{S}(\bar{c}^1, \bar{c}^2)$, where $\bar{c}^1 = c^1$, $\bar{c}^2 < c^2$, $(\bar{c}^1, \bar{c}^2) \in \mathcal{G}_1 \cup \mathcal{G}_2$ and $p(\bar{S}) \geq p(S)$ that dominates S.*

2. *Let $(c^1, c^2) \in \mathcal{G}_3$. Then S cannot be dominated by any knapsack S' satisfying $c^1 \geq w^1(S') \geq w^2(S') + c^1 - c^2$. Furthermore if S is dominated, then there exists a knapsack $\bar{S} \in \mathcal{S}(\bar{c}^1, \bar{c}^2)$, where $\bar{c}^1 < c^1$, $\bar{c}^2 = c^2$, $(\bar{c}^1, \bar{c}^2) \in \mathcal{G}_3$ and $p(\bar{S}) \geq p(S)$ that dominates S.*

Proof. Let $S \in \mathcal{S}(c^1, c^2)$ be a dominated knapsack and let $(c^1, c^2) \in \mathcal{G}_1 \cup \mathcal{G}_2$ with $c^1 \cdot c^2 \neq 0$. Then there exists a knapsack $\bar{S}$ such that $p(\bar{S}) \geq p(S)$, $w^1(\bar{S}) \leq c^1$ and $w^2(\bar{S}) \leq c^2$ with at least one strict inequality. Without loss of generality we may assume that $\bar{S} \in \mathcal{S}(\bar{c}^1, \bar{c}^2)$ where $\bar{c}^i = w^i(\bar{S})$ for $i = 1, 2$.
Since $S \in \mathcal{S}(c^1, c^2)$, we have that $\bar{S} \notin \mathcal{S}(c^1, c^2)$. Hence, $\bar{c}^1 < c^1$ or $\bar{c}^2 < c^2$.

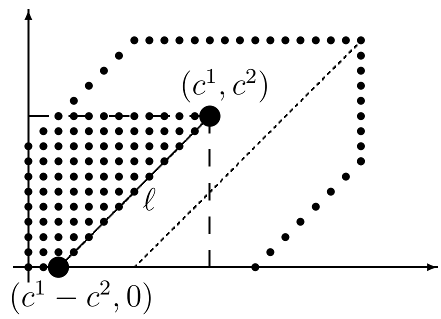

Figure 3: *Construction of the line segment ℓ in the proof of Theorem 4.8 for $(c^1, c^2) \in \mathcal{G}_2$.*

Assume that $\bar{c}^2 \geq \bar{c}^1 + c^2 - c^1$, i.e. the point $(\bar{c}^1, \bar{c}^2)$ does not lie underneath the line segment ℓ connecting the point (c^1, c^2) with $(0, c^2 - c^1)$, if $(c^1, c^2) \in G_1$, and $(c^1 - c^2, 0)$ if $(c^1, c^2) \in G_2$, respectively (see also Figure 3). Using Eq. (2) and Eq. (3) we conclude that $(\bar{c}^1, \bar{c}^2)$ must lie above the line segment.
Without loss of generality we may assume now that $\bar{c}^2 = c^2$ and $\bar{c}^1 < c^1$ since otherwise we can construct a new knapsack $\tilde{S} \in S(\tilde{c}^1, c^2)$ by applying Corollary 4.1, such that $p(\tilde{S}) > p(\bar{S}) \geq p(S)$ and $\tilde{c}^1 < c^1$.
First assume that $p(\bar{S}) > p(S)$. This implies that there exist two knapsacks $\bar{S}_1 \in \mathcal{S}(\bar{c}^1 + j, c^2)$ and $\bar{S}_2 \in \mathcal{S}(\bar{c}^1 + j + 1, c^2)$ for a fixed $j \in \{0, \ldots, c^1 - \bar{c}^1 - 1\}$ such that $p(\bar{S}_1) > p(\bar{S}^2)$ and hence, $\bar{S}_1$ would dominate $\bar{S}_2$ to the right. However, this is not possible in $G_1 \cup G_2$ due to Theorem 4.5 and Theorem 4.7, respectively.
Hence, we have that $p(\bar{S}) = p(S)$. Assume that $\bar{c}^1 = c^1 - 1$. This implies that $\bar{S}$ dominates S to the right which is not possible. Hence, $\bar{c}^1 \leq c^1 - 2$ and there exists another knapsack $\bar{S}_1 \in \mathcal{S}(\bar{c}^1 + 1, c^2)$ such that $p(\bar{S}_1) > p(\bar{S})$ since otherwise $\bar{S}_1$ would be dominated from the left by $\bar{S}$. But this implies once more that now $p(\bar{S}_1) > p(S)$, which is not possible in $G_1 \cup G_2$.
We conclude that $(\bar{c}^1, \bar{c}^2)$ must lie underneath the line segment ℓ. If $\bar{c}^1 < c^1$ we use once more Corollary 4.1 to construct a knapsack $\bar{S} \in \mathcal{S}(\bar{c}^1, \bar{c}^2)$, where $\bar{c}^1 = c^1$, $\bar{c}^2 < c^2$ is satisfied. Obviously, $\bar{S}$ dominates S. Since there is no dominance to the top in Sector $\mathcal{G}_3$ according to Theorem 4.4, we finally can assume that $(\bar{c}^1, \bar{c}^2) \in \mathcal{G}_1 \cup \mathcal{G}_2$. This proves (1.). By a similar line of argument, (2.) can be proven. $\square$

4.2 Lower Bound on the Cardinality of $\mathcal{Z}^{ND}$

In this second part we establish a strict lower bound on the cardinality of the non-dominated set $\mathcal{Z}^{ND}$ using the dominance relations in the different sectors.

Remark 4.9 *Without loss of generality we assume in the following that super-items are always included first in an optimal knapsack if there exist other items in D having the same profit value.*

Applying the rule stated in Remark 4.9 does not change the profit value of an optimal solution, but simplifies the reasoning in the following since tedious case differentiations can be omitted. We start with the knapsacks that belong to a basis.

Lemma 4.10 *Let $c' \in \{0, \ldots, n_R\}$ and $c^\star \in \{1, \ldots, n_U\}$. Then, $S' \in \mathcal{S}(c', 0)$ and $S^\star \in \mathcal{S}(0, c^\star)$ are efficient knapsacks of Problem (2-MP) with non-dominated criteria vectors*

$(p(S'), c', 0)$ *and* $(p(S^\star), 0, c^\star)$.

Proof. The efficiency of all knapsacks in the bases follows immediately from Lemma 3.2. $\square$

Next we state that all optimal knapsacks to Problem (2-KP$_=$) for constraints corresponding to integer points on the common boundary line between $\mathcal{G}_2$ and $\mathcal{G}_3$ are also efficient knapsacks of Problem (2-MP).

Lemma 4.11 *Let* $(c^1, c^2) \in \mathcal{G}$ *such that* $c^1 - c^2 = n_R - n_U$ *and* $S \in \mathcal{S}(c^1, c^2)$. *Then* $S \in \mathcal{N}^E$.

Proof. The lemma is an immediate consequence of Theorem 4.8 and Lemma 4.10. $\square$

To complete the second part of this section, we finally show that optimal knapsacks of Problem (2-KP$_=$) for constraint vectors (c^1, c^2) contained in the rectangle

$$\mathcal{Q} = \left\{(x, y) \in \mathbb{R}^2 : x \in \{0, 1, \ldots, n_R\}, y \in \{0, 1, \ldots, n_U\}\right\},$$

also correspond to efficient knapsacks of Problem (2-MP).

Theorem 4.12 *Let* $S \in \bigcup_{(c^1,c^2)\in\mathcal{Q}} \mathcal{S}(c^1, c^2)$. *Then* $S \in \mathcal{N}^E$.

Proof. We have to distinguish the three cases $S \in \mathcal{G}_i$ for $i \in \{1, 2, 3\}$. We give a proof for the case that $S \in \mathcal{G}_2$. The proofs for the remaining two other sectors follow the same ideas as the proof for Sector G_2.
If $(c^1, c^2) \in \mathcal{G}_2 \cap \mathcal{G}_3$, there is nothing to show according to Lemma 4.11. So let $(c^1, c^2) \in \mathcal{G}_2 \setminus \mathcal{G}_3$. We assume that there exists a knapsack $\bar{S}$ that dominates S. According to Theorem 4.8 there exists a well-defined index $\bar{c}^2 \in \{0, \ldots, c^2 - 1\}$ such that $\bar{S} \in \mathcal{S}(c^1, \bar{c}^2)$, and $(c^1, \bar{c}^2) \in \mathcal{G}_1 \cup \mathcal{G}_2$. Since $(c^1, c^2) \in \mathcal{G}_2$, also $(c^1, \bar{c}^2) \in \mathcal{G}_2$ must hold.
Next we show that we may assume that $\bar{S} \in \mathcal{S}(c^1, c^2 - 1)$. Otherwise there must exist a fixed index $i \in \{0, \ldots, c^2 - \bar{c}^2 - 1\}$ and $\bar{S}_i \in \mathcal{S}(c^1, \bar{c}^2 + i)$ and $\bar{S}_{i+1} \in \mathcal{S}(c^1, \bar{c}^2 + i + 1)$ such that $p(\bar{S}_i) > p(\bar{S}_{i+1})$. Setting $S = \bar{S}_{i+1}$ and $\bar{S} = \bar{S}_i$ implies that S is dominated by its neighbor $\bar{S}$ from below.
Now let $c = c^1 - c^2$ and let μ denote the number of super-items contained in $\bar{S}$. Note that since $(c^1, c^2) \in \mathcal{Q}$ it holds that $0 \leq \mu \leq c^2 - 1 \leq n_U - 1 < n_U$. According to Lemma 4.3 we know that $\bar{d}^c_k \geq \bar{d}^{c+1}_k \geq \bar{d}^c_{k+1}$ for all $k \in \{1, \ldots, n_D + n_U - 1\}$. This implies that S must contain at least μ but at most $\mu + 1$ super-items, assuming Remark 4.9 is valid. We

get:

$$
\begin{aligned}
p(\bar{S}) - p(S) &= r_{c+1} + \sum_{i=1}^{c^2-1} \bar{d}_i^{c+1} - \sum_{i=1}^{c^2} \bar{d}_i^{c} && (5)\\
&= r_{c+1} + \sum_{i=1}^{\mu} \tilde{d}_i^{c+1} + \sum_{i=1}^{c^2-1-\mu} d_i - \left(\sum_{i=1}^{\mu} \tilde{d}_i^{c} + \sum_{i=1}^{c^2-1-\mu} d_i + \bar{d}_{c^2}^{c} \right)\\
&= r_{c+1} + \sum_{i=1}^{\mu} r_{c+1+i} + \sum_{i=1}^{\mu} u_i - \left(\sum_{i=1}^{\mu} r_{c+i} + \sum_{i=1}^{\mu} u_i \right) - \bar{d}_{c^2}^{c}\\
&= r_{c+1+\mu} - \bar{d}_{c^2}^{c}\\
&= r_{c+1+\mu} - \max\left(d_{c^2-\mu}, r_{c+\mu+1} + u_{\mu+1}\right) && (6)\\
&< r_{c+1+\mu} - r_{c+1+\mu} \;=\; 0.
\end{aligned}
$$

Note that the element $d_{c^2-\mu}$ is not guaranteed to exist, but $\tilde{d}_{\mu+1}^{c}$ always exists since $\mu + 1 \leq n_U$. Hence, $p(\bar{S}) < p(S)$ and S cannot be dominated by $\bar{S}$. □

We summarize the last results in the following corollary.

Corollary 4.13 $\left|\mathcal{Z}^{ND}\right| \geq (n_U + 1) \cdot (n_R + 1) + n_D.$

Proof. The proof follows immediately from Lemma 4.10, Lemma 4.11 and Theorem 4.12. □

At the end of the next subsection we will show that the stated lower bound on the cardinality of the set of all non-dominated solutions is tight.

4.3 Omit the filtering step

The aim of this subsection is to show that we can omit the filtering step to remove dominated knapsacks at the end. We state necessary and sufficient conditions on the value of the profits of the items contained in R, U and D, respectively, which allow to decide whether an optimal knapsack for (2-KP$_=$) given by an element of the sequence $\{S^i\}_i$ stated in Corollary 4.1 is also an efficient knapsack of (2-MP).

We concentrate on Sector $\mathcal{G}_2$ in the following and give a detailed outline of the proofs implying our necessary and sufficient condition for this sector. Note that for $(c^1, c^2) \in \mathcal{G}_2$ the maximal number of super-items contained in a feasible knapsack for Problem (2-KP$_=$) is restricted to n_U. We start with the following lemma.

Lemma 4.14 *Let $(c^1, c^2) \in \mathcal{G}_2$ where $c^2 > n_U$ and let $S' \in \mathcal{S}(c^1, c^2 - 1)$ dominate $S \in \mathcal{S}(c^1, c^2)$. Then, S and S' contain all n_U super-items and $r_{c^1-c^2+n_U+1} \geq d_{c^2-n_U}$.*

Proof. Let μ and μ' denote the number of super-items that are contained in S and S', respectively. Applying Lemma 4.3 and Remark 4.9, we conclude that $\mu \in \{\mu', \mu' + 1\}$ whenever $\mu < n_U$. By applying the same reasoning of the proof for Theorem 4.12, it implies that $p(S') < p(S)$, and S' cannot dominate S. Hence, S must contain all n_U super-items.
Now, assume that S' does not contain all n_U super-items but only $n_U - 1$ although it

dominates S. Analyzing the elements of R, U, and D that are contained in S and S' leads to

$$0 \leq p(S') - p(S) = -u_{n_U} < 0.$$

Hence, also S' must contain all super-items. Using Eq. (6) from the proof of Theorem 4.12 and knowing that $\mu = n_U$ is maximal, we conclude that

$$0 \leq p(S') - p(S) = r_{c^1-c^2+1+n_U} - d_{c^2-n_U}.$$

Hence, $r_{c^1-c^2+1+n_U} \geq d_{c^2-n_U}$. □

Note that Lemma 4.14 is valid for $0 \leq c^1 - c^2 < n_R - n_U$, whereas for $c^1 - c^2 = n_R - n_U$, $S \in \mathcal{S}(c^1, c^2)$ is always an efficient solution according to Lemma 4.11. We also take care of this fact in the following lemma.

Lemma 4.15 *Let $(c^1, c^2) \in \mathcal{G}_2$ with $0 \leq c^1 - c^2 < n_R - n_U$ and $n_U < c^2 < n_U + n_D$, and let $S' \in \mathcal{S}(c^1, c^2 - 1)$ dominate $S \in \mathcal{S}(c^1, c^2)$. Then S dominates $\bar{S} \in \mathcal{S}(c^1, c^2 + 1)$.*

Proof. Let S, S' and $\bar{S}$ be given as defined above. According to Lemma 4.14, S must contain all n_U super-items and it holds that $r_{c+1+n_U} \geq d_{c^2-n_U}$. We assume first that $(c^1, c^2) \in \mathcal{G}_2 \setminus \mathcal{G}_1$, i.e. $c^1 > c^2$. We use Lemma 4.3 to deduce that $\bar{S} \in \mathcal{S}(c^1 - c^2 - 1, 0)$ must also contain all n_U super-items, since it contains an additional item compared to S. We conclude that

$$\begin{aligned} p(S) - p(\bar{S}) &= r_{c^1-c^2+n_U} - d_{c^2-n_U+1} \geq r_{c^1-c^2+n_U+1} - d_{c^2-n_U+1} \\ &\geq r_{c^1-c^2+n_U+1} - d_{c^2-n_U} \geq 0. \end{aligned}$$

If $(c^1, c^2) \in \mathcal{G}_2 \cap \mathcal{G}_1$, i.e. $c^1 = c^2$, $\bar{S}$ contains all $n_U - 1$ super-items with respect to its basis $\mathcal{S}(0, 1)$ according to Lemma 4.14. Adapting the chain of inequalities stated above to this special case shows that S dominates $\bar{S}$. □

We are now able to derive the main result for $\mathcal{G}_2$.

Theorem 4.16 *Let $(c^1, c^2) \in \mathcal{G}_2$ with $0 \leq c^1 - c^2 \leq n_R - n_U - 1$ and $n_U + 1 \leq c^2 \leq n_U + n_D$, and let $S \in \mathcal{S}(c^1, c^2)$. Then the following statements are equivalent:*

(A) *$S \notin \mathcal{N}^E$.*

(B) *S is dominated by $S' \in \mathcal{S}(c^1, c^2 - 1)$.*

(C) *$r_{n_U+c^1-c^2+1} \geq d_{c^2-n_U}$.*

Proof. '(B)⇒(A)' is obviously true and the proof for '(B)⇒(C)' was already shown in Lemma 4.14. We prove the following implications:

'(A)⇒(B)'. Let $S \in \mathcal{S}(c^1, c^2)$ be dominated by $\bar{S} \in \mathcal{S}(\bar{c}^1, \bar{c}^2)$ with $\bar{c}^1 \leq c^1$ and $\bar{c}^2 \leq c^2$ with at least one strict inequality. According to Theorem 4.8 we may assume that $(\bar{c}^1, \bar{c}^2) \in \mathcal{G}_2$, $\bar{c}^1 = c^1$ and $\bar{c}^2 < c^2$. Assume that S is not dominated by $S' \in \mathcal{S}(c^1, c^2 - 1)$, i.e. $\bar{c}^2 < c^2 - 1$ and $p(S') < p(S) \leq p(\bar{S})$. Then, there must exist a fixed index $i \in \{0, \ldots, c^2 - \bar{c}^2 - 2\}$ and $\bar{S}_i \in \mathcal{S}(c^1, \bar{c}^2 + i)$ and $\bar{S}_{i+1} \in \mathcal{S}(c^1, \bar{c}^2 + i + 1)$ such that $p(\bar{S}_i) > p(\bar{S}_{i+1})$ holds. But this implies that $\bar{S}_{i+1}$ is dominated by its neighbor $\bar{S}_i$ from below. By applying Lemma 4.15, also S must be dominated by its neighbor from below, i.e. S' dominates S, which contradicts our assumption.

'(C)⇒(B)'. Since

$$r_{c^1-c^2+n_U} + u_{n_U} \;>\; r_{c^1-c^2+n_U} \;\geq\; r_{c^1-c^2+n_U+1} \;\geq\; d_{c^2-n_U},$$

S consists of all first $c^1 - c^2$ items of R, all n_U super-items and the first $c^2 - n_U$ items of D. Furthermore, it holds that

$$r_{c^1-c^2+n_U+1} + u_{n_U} \;>\; r_{c^1-c^2+n_U+1} \;\geq\; d_{c^2-n_U},$$

and S' contains at most $c^2 - n_U - 1$ items of D, at most n_U super-items and the first $c^1 - c^2 + 1$ items of R. But since $(c^2 - n_U - 1) + n_U = c^2 - 1$ and this is the number of elements that are added to the knapsack in basis $\mathcal{S}(c^1 - c^2, 0)$, S' must contain exactly the above mentioned items. We conclude that

$$p(S') - P(S) \;=\; r_{c^1-c^2+n_U+1} - d_{c^2-n_U} \;\geq\; 0.$$

□

In Theorem 4.16 we have proven a necessary and sufficient condition for the efficiency of an optimal knapsack $S \in \mathcal{S}(c^1, c^2)$ where $(c^1, c^2) \in \mathcal{G}_2$, which supersedes the filtering for dominated solutions. Given the sequence $\{S^i\}_i$ stated in Corollary 4.1 we stop calculating the elements of this sequence when $r_{c^1-c^2+n_U+1} \geq d_{c^2-n_U}$ is satisfied for the first time. Starting from this element all remaining knapsacks of the sequence will be dominated by their neighbor from below. Hence, an additional filtering is no longer needed.

By a similar line of argument as used in Lemma 4.14 and Lemma 4.15, it is possible to prove similar results of Theorem 4.16 for $\mathcal{G}_1$ and $\mathcal{G}_3$.

Theorem 4.17 *Let $(c^1, c^2) \in \mathcal{G}_1$ with $1 \leq c^2 - c^1 \leq n_U$ and $n_U + 1 \leq c^2 \leq n_U + n_D$, and let $S \in \mathcal{S}(c^1, c^2)$. Then the following statements are equivalent:*

(A) *$S \notin \mathcal{N}^E$.*

(B) *S is dominated by $S' \in \mathcal{S}(c^1, c^2 - 1)$.*

(C) *$r_{n_U+c^1-c^2+1} \geq d_{c^2-n_U}$.*

For Sector $\mathcal{G}_3$ we get:

Theorem 4.18 *Let $(c^1, c^2) \in \mathcal{G}_3$ with $n_R - n_U + 1 \leq c^1 - c^2 \leq n_R$ and $n_R + 1 \leq c^1 \leq n_R + n_D$, and let $S \in \mathcal{S}(c^1, c^2)$. Then the following statements are equivalent:*

(A) *$S \notin \mathcal{N}^E$.*

(B) *S is dominated by $S' \in \mathcal{S}(c^1 - 1, c^2)$.*

(C) *$u_{n_R-c^1+c^2+1} \geq d_{c^1-n_R}$.*

The pseudo-code of the resulting algorithm for $\mathcal{G}_2$ is described in Algorithm 1. We omit the pseudo-code for $\mathcal{G}_1$ and $\mathcal{G}_3$. For the sake of the explanation, we assume that the non-dominated vector $(0, 0, 0)$ is computed at sector $\mathcal{G}_2$ in Algorithm 1.

Note that the last results imply that the lower bound on the number of non-dominated solutions that was stated in Corollary 4.13 is tight. Assume that for a given instance of (2-MP) we have that $\min\{r_{n_R}, u_{n_U}\} > d_1$. Then the third criterion stated in the Theorems 4.16 to 4.18 for the different sectors immediately implies that the bound is tight. If by contrast $\max\{r_{n_R}, u_{n_U}\} < d_{n_D}$ the stated criteria imply that solving (2-KP$_{\leq}$) for any $(c^1, c^2) \in \mathcal{G}$ will lead to a non-dominated solution of (m-MP).

Algorithm 1 Greedy algorithm for Problem (2-MP) in Sector $\mathcal{G}_2$

input: An instance of Problem (2-MP).
output: The non-dominated set in sector $\mathcal{G}_2$ ($Z_{\mathcal{G}_2}$).

Pre-processing step (see Section 2).
$b \leftarrow 0$
$Z_{\mathcal{G}_2} \leftarrow \emptyset$
for $c = 1$ to $n_R - n_U$ **do**
  $b \leftarrow b + r_c$
  $Z_{\mathcal{G}_2} \leftarrow Z_{\mathcal{G}_2} \cup \{(c, 0, b)\}$
  $p \leftarrow b$
  $i \leftarrow 1$
  $j \leftarrow 1$
  repeat
    if $d_j < r_{c+i} + u_i$ **then**
      $p \leftarrow p + r_{c+i} + u_i$
      $Z_{\mathcal{G}_2} \leftarrow Z_{\mathcal{G}_2} \cup \{(c + i + j - 1, i + j - 1, p)\}$
      $i \leftarrow i + 1$
    else
      $p \leftarrow p + d_j$
      $Z_{\mathcal{G}_2} \leftarrow Z_{\mathcal{G}_2} \cup \{(c + i + j - 1, i + j - 1, p)\}$
      $j \leftarrow j + 1$
    end if
  until $r_{n_U+c+1} \geq d_{i+j-1-n_U}$ and $n_U + 1 \leq i + j - 1 \leq n_U + n_D$
end for
return $Z_{\mathcal{G}_2}$

4.4 Connectedness

Corollary 4.1 introduces an important result in terms of *connectedness* of efficient knapsacks, for which only negative results are known [4]. We define a graph where the nodes represent the efficient knapsacks and edges are introduced between all pairs of nodes that are adjacent with respect to the following definition of *k-change* neighborhood:

Definition 4.19 (k-change neighborhood) *Two knapsacks are neighbors with respect to the k-change neighborhood if and only if one knapsack can be obtained from the other either by either adding or removing at most k items.*

We say that the efficient set of Problem (2-MP) is connected if and only if the corresponding graph is connected. In our case, we state the following result for the 2-change neighborhood.

Corollary 4.20 *There exists a set of efficient knapsacks of Problem (2-MP) that is connected with respect to the 2-change neighborhood and its image in the criteria space coincides with the non-dominated set.*

Corollary 4.20 only states that a subset of the set of efficient knapsacks is connected but not the complete set itself. However, this property applies to the complete set of efficient knapsacks, if we consider also the following extended definition of neighborhood:

Definition 4.21 (k-exchange neighborhood) *Two knapsacks are neighbors with respect to the k-exchange neighborhood if and only if one knapsack can be obtained from the other by exchanging and adding or removing at most k items.*

Note that in a k-change neighborhood it is only allowed to either add or remove k items to or from a knapsack, respectively, while in a k-exchange neighborhood, we can exchange a number of items and either add or remove another fixed number. For example, exchanging a super-item by another one is considered as a 2-exchange, as well as an exchange of a super-item by an item contained in D. In this case we have to exchange one item and either add or remove another one. By the definition of a k-exchange, a k-change is always a k-exchange, but not necessarily the other way round.

Theorem 4.22 *The set of efficient knapsacks of Problem (2-MP) is connected with respect to the* 2*-exchange neighborhood.*

Proof. According to Corollary 4.20 it suffices to show that for an efficient $S \in S(c^1, c^2)$ and an alternative efficient knapsack $\bar{S} \in S(c^1, c^2)$, there exist a finite sequence of efficient knapsacks starting from S and ending by $\bar{S}$, such that all knapsacks of this sequence are contained in $S(c^1, c^2)$ and subsequent knapsacks are neighbors with respect to the 2-exchange.

For the case that S is contained in a basis $\mathcal{S}(c, 0)$ (or $\mathcal{S}(0, c)$), alternative optima exist if and only if the profit value r_c (or u_c) is not unique, i.e. there exists $\tilde{c} \in \{c+1, \ldots, n_R\}$ (or $\tilde{c} \in \{c+1, \ldots, n_U\}$) such that $r_c = \ldots = r_{\tilde{c}}$ (or $u_c = \ldots = u_{\tilde{c}}$). But obviously all the resulting efficient knapsacks are connected with respect to a 1-exchange neighborhood, since we can exchange items having the same profit value one by one to construct an appropriate sequence.
Now let $\min\{c^1, c^2\} > 0$. If for efficient $S \in \mathcal{S}(c^1, c^2)$, there exist another alternative efficient knapsack $\bar{S} \in \mathcal{S}(c^1, c^2)$, a similar reasoning as in the case of a basis applies. Either a number of profit values in the sequences R and U are not unique or the same property applies to the the profit value $\bar{d}_{c^2}^{c^1-c^2}$, if $c^2 \leq c^1$, or $\bar{d}_{c^1}^{c^2-c^1}$, if $c^1 < c^2$ of the sequence $\bar{D}^{c^1-c^2}$ or $\bar{D}^{c^2-c^1}$. But all the resulting alternative knapsacks are connected with respect to a 2-exchange neighborhood, since at worst we have to exchange a super-item by another super-item to construct another alternative efficient knapsack. Hence, an appropriate sequence, starting from S and ending by $\bar{S}$ such that subsequent knapsacks are neighbors with respect to the 2-exchange neighborhood can always be found within the set $\mathcal{S}(c^1, c^2)$. $\square$

It is worth mentioning that the (proof of the) connectedness of the set of efficient knapsack is not based on the connectedness of supported efficient knapsacks as shown in [3], but that the proof is constructive.

4.5 Experimental Results

To verify the efficiency of our approach in practice, we implemented it in C and tested it on a set of randomly generated instances.[1] We generated 100 instances for each of the

[1] The code is available at `http://eden.dei.uc.pt/~paquete/mpt.c`.

n	CPU-time (in secs.)		$\lvert\mathcal{Z}^{ND}\rvert$	
	avg.	std.	avg.	std.
1 000	0.00	0.00	182 329	42 472
2 500	0.01	0.00	1 116 855	256 644
5 000	0.04	0.01	4 478 076	1 192 776
7 500	0.09	0.02	9 786 613	2 388 547
10 000	0.15	0.02	18 211 143	3 919 746
25 000	0.97	0.19	117 716 464	26 611 280
50 000	3.80	0.79	448 329 030	110 881 051
75 000	8.10	2.12	980 416 031	275 287 540
100 000	14.73	3.62	1 766 044 758	469 409 790
250 000	93.81	22.74	11 267 090 109	3 036 879 464
500 000	381.11	89.58	44 872 436 605	11 436 272 293
750 000	877.49	178.36	102 117 237 786	23 926 930 515
1000 000	1542.83	333.52	179 661 247 582	41 598 896 050

Table 1: Average (avg.) and standard deviation (std.) of CPU-time in seconds taken by the greedy algorithm and the size of the non-dominated set for randomly generated instances of Problem (2-MP).

sizes $n = \{10 \times 10^i, 25 \times 10^i, 50 \times 10^i, 75 \times 10^i\}$ with $i = 2, 3, 4$. The profit values are positive integers uniformly distributed in the range $[1, 11]$; we choose a small range of profit values to avoid number overflow in large instances. In order to generate values for n_R, n_U and n_D, we first generated three real numbers randomly and uniformly distributed in the range $[0, 1]$; then, we normalized these values with respect to their sum; finally, we multiplied each normalized value by n to obtain n_R, n_U and n_D, respectively. We ran our implementation on an Intel Core 2 Duo 2.33Ghz, 4MB cache L2, 4GB RAM, with Windows Vista 32 bits, compiler MSVC 2008.

Table 4.5 shows the average and standard deviation of CPU-time in seconds taken by our greedy algorithm, as well as the cardinality of the non-dominated set for the randomly generated instances. They clearly indicate that our approach can perform very fast.

5 A greedy algorithm for Problem (2-KP$_\leq$)

Based on the results for solving Problem (2-MP), we can derive an efficient algorithm to solve Problem (2-KP$_\leq$). Note that if Problem (2-KP$_\leq$) is feasible, i.e. c^1 and c^2 are chosen to be non-negative, an optimal knapsack $\bar{S}$ to this problem is contained in the efficient set of Problem (2-MP). Obviously, for such a knapsack it holds that $f(\bar{S}) \geq f(S)$ for all $S \in \mathcal{N}^E$ where $w^j(S) \leq c^j$ and $w^j(\bar{S}) \leq c^j$, $j = 1, 2$, respectively. Hence, the results of Section 4 can be used to solve Problem (2-KP$_\leq$).

We will show that we only need to consider one of the sequences $\{S^i\}_i$ starting from an efficient knapsack contained in an appropriate chosen basis. The superscripts of the optimal knapsacks in the following theorems indicate which element of the sequence $\{S^i\}_i$ has to be calculated.

To apply the results of Section 4 we need to consider an additional partition of each

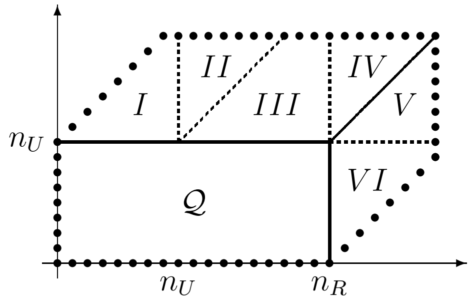

Figure 4: *Illustration of the partition of $\mathcal{G}$ for theorems 5.1, 5.2 and 5.3.*

sector $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ as illustrated in Figure 4. We start with Sector $\mathcal{G}_2$ and recall that there is no dominance to the right in this sector and that an efficient knapsack is dominated if and only if it is dominated by its neighbor from below.

Theorem 5.1 *Let $(c^1, c^2) \in \mathcal{G}_2$.*

1. *If $(c^1, c^2) \in \mathcal{Q}$ or $c^1 - c^2 = n_R - n_U$, then $S^{c^2} \in \mathcal{S}(c^1, c^2)$ is an optimal knapsack of Problem (2-KP$_{\leq}$).*

2. *Let $(c^1, c^2) \notin \mathcal{Q}$, $c^1 - c^2 \neq n_R - n_U$ and $c^1 \leq n_R$. If there exists an index j such that $j = \min\{i \in \{0, \ldots, (c^2 - 1) - n_U\} : d_{c^2-i-n_U} > r_{n_U+c^1-c^2+i+1}\}$, then $S^{c^2-j} \in \mathcal{S}(c^1, c^2 - j)$ is an optimal knapsack of Problem (2-KP$_{\leq}$). Otherwise $S^{n_U} \in \mathcal{S}(c^1, n_U)$ is optimal.*

3. *Let $(c^1, c^2) \notin \mathcal{Q}$, $c^1 - c^2 \neq n_R - n_U$ and let $c^1 > n_R$. If there exists an index j such that $j = \min\{i \in \{0, \ldots, n_R - n_U - c^1 + c^2 - 1\} : d_{c^2-i-n_U} > r_{n_U+c^1-c^2+i+1}\}$, then $S^{c^2-j} \in \mathcal{S}(c^1, c^2 - j)$ is an optimal knapsack of Problem (2-KP$_{\leq}$). Otherwise $S^{c^1-n_R+n_U} \in \mathcal{S}(c^1, c^1 - n_R + n_U)$ is optimal.*

Proof. In the following, we prove the three cases above:

(1.) Let $(c^1, c^2) \in \mathcal{Q}$ and assume that $S^{c^2} \in \mathcal{S}(c^1, c^2)$ is not optimal for Problem (2-KP$_{\leq}$). Then there must exist another feasible knapsack $S \notin \mathcal{S}(c^1, c^2)$ satisfying $p(S) > p(S^{c^2})$ and $w^j(S) \leq w^j(S^{c^2})$, $j = 1, 2$ with at least one strict inequality. But this implies that S dominates S^{c^2}, which is not possible due to Theorem 4.12. Hence, S^{c^2} must be optimal. A similar reasoning in combination with Lemma 4.11 can be applied for the case that $c^1 - c^2 = n_R - n_U$.

(2.) Let $(c^1, c^2) \notin \mathcal{Q}$ and $c^1 - c^2 \neq n_R - n_U$, but $c^1 \leq n_R$. Furthermore, let S be an optimal knapsack of Problem (2-KP$_{\leq}$). Without loss of generality we may assume that $S \in \mathcal{N}^E$ and that there exist $\bar{c}^1 \in \{0, \ldots, c^1\}$ and $\bar{c}^2 \in \{0, \ldots, c^2\}$ such that $S \in \mathcal{S}(\bar{c}^1, \bar{c}^2)$ where either $\bar{c}^1 = c^1$ or $\bar{c}^2 = c^2$ due to the Equations (2), (3) and (4). Since there cannot be dominance to the right in the Sectors $\mathcal{G}_1$ and $\mathcal{G}_2$ according to Theorem 4.5 and Theorem 4.7 we conclude that $\bar{c}^1 = c^1$. Now, define $j = \min\{i \in \{0, \ldots, (c^2 - 1) - n_U\} : d_{c^2-i-n_U} > r_{n_U+c^1-c^2+i+1}\}$, if the minimum exists. Otherwise let $j = \infty$. Note that the index $n_U + c^1 - c^2 + i + 1$ is well-defined for all $i \in \{0, \ldots, (c^2 - 1) - n_U\}$, since obviously $n_U + c^1 - c^2 + i + 1 \geq 1$ and further

$$n_U + c^1 - c^2 + i + 1 \;\leq\; n_U + (n_R) - c^2 + (c^2 - n_U - 1) + 1 \;=\; n_R.$$

If $j = 0$, $S^{c^2} \in \mathcal{S}(c^1, c^2)$ must be optimal, since S^{c^2} is efficient due to Theorem 4.16 and hence, $p(S)$ must be maximal for all knapsacks S satisfying $w^j(S) \leq c^j$ for $j = 1, 2$. Now, assume that $1 \leq j < \infty$ and let $S^{c^2-j} \in \mathcal{S}(c^1, c^2 - j)$. Theorem 4.16 states that a knapsack $S \in \mathcal{S}(c^1, \tilde{c}^2)$ is dominated by its neighbor from below whenever $\tilde{c}^2 \in \{c^2 - j + 1, \ldots, c^2\}$. But this implies that $p(S) \leq p(S^{c^2-j})$ for all $S \in \mathcal{S}(c^1, \tilde{c}^2)$. Since S^{c^2-j} is efficient according to the same theorem, it follows that $p(S) < p(S^{c^2-j})$ for all $S \in \mathcal{S}(c^1, \tilde{c}^2)$ whenever $\tilde{c}^2 \in \{0, \ldots, c^2 - j - 1\}$. This shows, that $p(S^{c^2-j})$ is optimal for Problem (2-KP$_\leq$). If $j = \infty$ this implies that all knapsacks $S \in \mathcal{S}(c^1, \bar{c}^2)$ are dominated by its neighbor from below for $\bar{c}^2 \in \{n_U + 1, \ldots, c^2\}$. Let $S^{n_U} \in \mathcal{S}(c^1, n_U)$. Since S^{n_U} is contained in $\mathcal{Q}$, S^{n_U} is efficient by Theorem 4.12, and hence it must be optimal for Problem (2-KP$_\leq$).

(3.) The proof for the case that $c^1 > n_R$ follows the same line of argument as the proof for the case $c^1 \leq n_R$. Note that now Lemma 4.11 has to be used instead of Theorem 4.12 and that the index $c^2 - i - n_U$ is well-defined for all $i \in \{0, \ldots, n_R - n_U - c^1 + c^2 - 1\}$ since $c^2 - i - n_U \leq c^2 - n_U \leq n_D$ and further

$$c^2 - i - n_U \geq c^2 - (n_R - n_U - c^1 + c^2 - 1) - n_U = c^1 - n_R + 1 > 1.$$

□

For $(c^1, c^2) \in \mathcal{G}_1$ it is easy to varify that we can find an optimal knapsack S to Problem (2-KP$_\leq$) such that $S \in \mathcal{S}(c^1, \bar{c}^2)$ and $(c^1, \bar{c}^2)$ is also contained in $\mathcal{G}_1$, whenever $c^1 \leq n_U$ holds. For the case that $c^1 > n_U$, it may happen that $(c^1, \bar{c}^2)$ is no longer contained in $\mathcal{G}_1$ but in $\mathcal{G}_2$.

Theorem 5.2 *Let $(c^1, c^2) \in \mathcal{G}_1$.*

1. *If $(c^1, c^2) \in \mathcal{Q}$, then $S^{c^1} \in \mathcal{S}(c^1, c^2)$ is an optimal knapsack of Problem (2-KP$_\leq$).*

2. *Let $(c^1, c^2) \notin \mathcal{Q}$ and let $c^1 \leq n_U$. If there exists an index j such that $j = \min\{i \in \{0, \ldots, (c^2 - 1) - n_U\} : d_{c^2-i-n_U} > r_{n_U+c^1-c^2+i+1}\}$, then $S^{c^1} \in \mathcal{S}(c^1, c^2 - j)$ optimally solves Problem (2-KP$_\leq$). Otherwise $S^{c^1} \in \mathcal{S}(c^1, n_U)$ is optimal.*

3. *Let $(c^1, c^2) \notin \mathcal{Q}$ and let $c^1 > n_U$ If there exists an index j such that $j = \min\{i \in \{0, \ldots, c^2 - c^1 - 1\} : d_{c^2-i-n_U} > r_{n_U+c^1-c^2+i+1}\}$, then $S^{c^1} \in \mathcal{S}(c^1, c^2 - j)$ is an optimal knapsack of Problem (2-KP$_\leq$). Otherwise there exists $\bar{c}^2 \in \{n_U, \ldots, c^1\}$, such that $(c^1, \bar{c}^2) \in \mathcal{G}_2$ and $S^{\bar{c}^2} \in \mathcal{S}(c^1, \bar{c}^2)$ is optimal for Problem (2-KP$_\leq$).*

Proof. The proofs of the three cases are similar to the proofs the corresponding statements in Theorem 5.1. In the last two cases, the minimum may not exist since either $c^1 = c^2$ or all knapsacks $S \in \mathcal{S}(c^1, \tilde{c}^2)$ with $\tilde{c}^2 \in \{c^1 + 1, \ldots, c^2\}$ are dominated by their neighbors from below. But both implies that there must exist an efficient knapsack $S \in \mathcal{S}(c^1, \bar{c}^2)$, where $(c^1, \bar{c}^2) \in \mathcal{G}_2$ and $\bar{c}^2 \in \{n_U, \ldots, c^1\}$ holds. □

For the Sector $\mathcal{G}_3$ we find similar results compared to the other two sectors.

Theorem 5.3 *Let $(c^1, c^2) \in \mathcal{G}_3$.*

1. *If $(c^1, c^2) \in \mathcal{Q}$ or $c^1 - c^2 = n_R - n_U$, then $S^{c^2} \in \mathcal{S}(c^1, c^2)$ is an optimal knapsack of Problem (2-KP$_\leq$).*

Step	Greedy Solution		Optimal Solution	
	profit value	171	174	profit value
1	L	22	22	L
2	$L+R$	$16+12=28$	$16+12=28$	$L+R$
3	$L+R$	$9+5=14$	8	RL
4	D	40	40	D
5	D	39	39	D
6	$R+LU$	28	$9+28=37$	$L+RU$

Table 2: Steps of the greedy algorithm to an instance of Problem (3-KP$_=$).

2. *Let* $(c^1, c^2) \notin \mathcal{Q}$, $c^1 - c^2 \geq n_R - n_U + 1$ *and let* $c^2 \leq n_U$. *If there exists an index* j *such that* $j = \min\{i \in \{0, \ldots, (c^1 - 1) - n_R\} : d_{c^1-i-n_R} > u_{n_R-c^1+c^2+i+1}\}$, *then* $S^{c^2} \in \mathcal{S}(c^1 - j, c^2)$ *is an optimal knapsack of Problem (2-KP*$_\leq$*). Otherwise* $S^{c^2} \in \mathcal{S}(n_R, c^2)$ *is optimal.*

3. *Let* $(c^1, c^2) \notin \mathcal{Q}$, $c^1 - c^2 \geq n_R - n_U + 1$ *and let* $c^2 > n_U$. *If there exists an index* j *such that* $j = \min\{i \in \{0, \ldots, n_U - n_R + c^1 - c^2 - 1\} : d_{c^1-i-n_R} > r_{n_R-c^1+c^2+i+1}\}$, *then* $S^{c^2} \in \mathcal{S}(c^1 - j, c^2)$ *is an optimal knapsack of Problem (2-KP*$_\leq$*). Otherwise* $S^{c^2} \in \mathcal{S}(n_R - n_U, c^2)$ *is optimal.*

The Theorems 5.1 to 5.3 show that we can determine an optimal knapsack to Problem (2-KP$_\leq$) by calculating a fixed number of elements of a sequence $\{S^i\}_i$ used in Corollary 4.1 for a knapsack S^0 contained in an approriate chosen basis. Therefore, the algorithm for solving this problem has the same time complexity of the greedy algorithm for (2-KP$_=$).

6 Discussion and conclusions

In this article we presented efficient algorithms to solve interesting special cases of three hard optimization problems within a polynomial amount of time. In particular, for the case of Problem (2-MP), our implementation of the algorithm is able to find the complete non-dominated set in half an hour for instances with one million of items, which corresponds to more than 400 millions distinct solutions on average.

Note that the proposed greedy algorithm for Problem (2-KP$_=$) may suggest that this approach can be easily extended to larger dimensions/criteria, depending only on the way the items are partitioned in the pre-processing step and using a similar decomposition as in (1). Unfortunately, it is not the case, not even for the Problem (3-KP$_=$), as shown in the following example.

Example 6.1 Consider an instance of Problem (3-KP$_=$). We partition the set of items according to their weights $(w^1(s), w^2(s), w^3(s))$ for all items s and obtain seven different sets where all elements in a set have the weights $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$, respectively. We denote these sets by R, L, U, RL, RU, LU and D, respectively. Consider the following partitoning of items: $R = (12,5,4)$, $L = (22,16,9)$, $U = (7,6,5)$, $RL = (8,5,4)$, $RU = (28,8,7)$, $LU = (24,9,7)$, $D = (40,39,20)$, and a constraint given by $c = (5, 6, 3)$, it holds that

$$\begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The left-hand side of Table 2 shows which solutions would be included by the greedy approach using the above decomposition while the right-hand-side shows the corresponding decomposition of the optimal solution. Choosing the last element from L in step 3 for the greedy solution is "the wrong choice" since it blocks all elements from UR for further inclusion in better solutions.

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